Repetition Versus Noiseless Quantum Codes For Correlated Errors

Carlo Cafaro* and Stefano Mancini†
Dipartimento di Fisica, Università di Camerino, I-62032 Camerino, Italy

We study the performance of simple quantum error correcting codes with respect to correlated noise errors characterized by a finite correlation strength μ. Specifically, we consider bit flip (phase flip) noisy quantum memory channels and use repetition and noiseless quantum codes. We characterize the performance of the codes by means of the entanglement fidelity $\mathcal{F}(\mu, p)$ as function of the error probability p and degree of memory μ. Finally, comparing the entanglement fidelities of repetition and noiseless quantum codes, we find a threshold $\mu^*(p)$ for the correlation strength that allows to select the code with better performance.

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I. INTRODUCTION

Decoherence is the most important obstacle in quantum computing. It causes a quantum computer to lose its quantum properties destroying its performance advantages over a classical computer. Therefore, in order to maintain quantum coherence in any computing system, it is important to remove the unwanted entanglement with its noisy environment. The unavoidable interaction between the open quantum computing system and its environment corrupts the information stored in the system and causes computational errors that may lead to wrong outputs. In general, environments may be very complex systems characterized by many uncontrollable degrees of freedom. The principal task of quantum error correction (QEC, [1, 2]) is to tackle this decoherence problem. For a comprehensive introduction to quantum error correction, we refer to the work presented in [3]. In summary, there exists two strategies to defend quantum coherence of a processing against environmental noise. The first strategy is that of *quantum error correcting codes* (QECC) [4, 5] where, in analogy to classical information theory, quantum information is stabilized by using redundant encoding and measurements. This is also known as an active strategy. The second strategy is known as *noiseless quantum coding* (also known as error avoiding quantum coding or decoherence free subspaces (DFSs)) [6–10]. This is a passive strategy where quantum information is stabilized by exploiting symmetry properties of the environment-induced noise for suitable redundant encoding.

The formal mathematical description of the qubit-environment interaction is usually given in terms of quantum channels. When noise errors act independently on each qubit, we talk about memoryless (noisy quantum) channels and independent error models. Instead, when noise errors do not affect qubits independently but correlations between errors on different qubits must be taken into consideration, we talk about memory (noisy quantum) channels and correlated error models. Correlations between errors may be considered either temporally over each use of a single channel, or spatially between uses of many parallel channels. QECC were developed under the assumption of i.i.d. (identically and independently distributed) errors. Recent studies on the performance of quantum codes for memory channels appear in [11, 12]. In Ref. [13], the performance of some codes (CSS codes and n-qubit repetition code) for spatially-correlated errors were studied and characterized by means of the lowest order temporal expansion of the fidelity of the density operator representing the quantum register after a single application of error correction.

In this Letter, we study the performance of simple quantum error correcting codes in the presence of correlated (classical-like) noise error models characterized by a correlation strength. Specifically, we consider bit flip (phase flip) noisy quantum memory channels and use repetition and noiseless quantum codes. Although the error models considered are classical-like, we can gain useful insights for extending error correction techniques to fully quantum correlated error models. We characterize the performance of such error correcting codes by means of the entanglement fidelity [14] as function of the error probability and degree of memory. Finally, comparing the entanglement fidelities of repetition codes and noiseless quantum codes, we find a threshold for the correlation strength that allows to select the code with better performance.

The layout of this Letter is as follows. In Section II, the algorithmic structure of a basic quantum error correcting code and a brief description of independent and correlated error models are presented. In Section III, we introduce a simple error model in the presence of correlated errors. Specifically, we consider bit flip (or phase flip) noisy quantum

*Electronic address: carlo.cafaro@unicam.it
†Electronic address: stefano.mancini@unicam.it

memory channels. The performance of quantum error correcting codes is quantified by means of the entanglement fidelity $\mathcal{F}_{RC}^{(n)}(\mu,\, p)$ (RC = repetition code; n is the length of the code) as function of the error probability p and degree of memory μ. In Section IV, we use odd and even-length error avoiding quantum error correcting codes and characterize the performance of the codes by means of the entanglement fidelity $\mathcal{F}_{DFS}^{(n)}(\mu,\, p)$ [15]). In Section V, we briefly discuss the existence of threshold values for the correlation strength $\mu^{*}(p)$ that allows to select, for a fixed value of the dimension of the coding space and the error probability p, the quantum error correcting code with better performance and we present our final remarks.

II. QUANTUM ERROR CORRECTION AND ERROR MODELS

In this Section, we present the algorithmic structure of a basic quantum error correcting code and briefly describe independent and correlated error models in quantum computation.

A. Algorithmic Structure of Quantum Error Correction

Characterizing the Error Model. We may deal with error models where errors occur on single qubits independently (independent error models). However, we may also consider models in which errors do not affect qubits independently. In this case we have to take into account any correlation between errors on different qubits (correlated error model). In any case, the error model is completely described by the Kraus operators $\{A_j\}$ defining the quantum error operation Λ. The trace-preserving superoperator Λ is described as follows,

$$\rho \overset{\text{def}}{=} |\psi\rangle\langle\psi| \overset{\Lambda}{\longrightarrow} \Lambda(\rho) \overset{\text{def}}{=} \sum_j A_j |\psi\rangle\langle\psi| A_j^{\dagger}. \tag{1}$$

Introducing Redundancy by Choosing the Encoding. The basic idea behind the redundancy is that even when errors corrupt some of the qubits in a codeword $|\psi_{\text{enc}}\rangle$ [1], the remaining qubits contain enough information so that the logical qubit $|\psi\rangle$, representing the quantum information to be transmitted through the quantum noisy communication channel Λ, can be recovered. In quantum error correction, encoding is implemented via a unitary operator U_{enc} that acts on the state we wish to encode, tensored with an ancilla of some fixed number of qubits in some specified initial state. The goal is to choose the encoding operation U_{enc} in such a way that the behavior of these transformed errors allows us to find a recovery operation $\mathcal{R}$ that gives back $|\psi\rangle\langle\psi| \otimes \rho_{\text{noise}}$. The encoding operation is described as follows,

$$|\psi\rangle \overset{\text{tensor product}}{\longrightarrow} |\psi\rangle \otimes |00...0\rangle \overset{U_{\text{enc}}}{\longrightarrow} U_{\text{enc}}(|\psi\rangle \otimes |00...0\rangle) \overset{\text{def}}{=} |\psi_{\text{enc}}\rangle\,. \tag{2}$$

In other words, we consider the tensor product between the logical qubit $|\psi\rangle$ and the ancilla qubit $|00...0\rangle$, and then we encode $|\psi\rangle \otimes |00...0\rangle$ via a unitary operator U_{enc}, obtaining $U_{\text{enc}}(|\psi\rangle \otimes |00...0\rangle) \overset{\text{def}}{=} |\psi_{\text{enc}}\rangle$.

Finding a Procedure for Error Recovery. The encoding operation U_{enc} can be seen as a way of transforming the encoded errors A_j' so that their action on the codeword states $|\psi_{\text{enc}}\rangle$ is recoverable. Before encoding, the quantum error operation Λ is defined as in (1. After encoding, the new quantum error operation A' is defined as follows,

$$\rho_{\text{enc}} \overset{\text{def}}{=} |\psi_{\text{enc}}\rangle\langle\psi_{\text{enc}}| \overset{\Lambda'}{\longrightarrow} \Lambda'(\rho_{\text{enc}}) \overset{\text{def}}{=} \sum_j A_j' |(\psi_{\text{enc}})\rangle\langle\psi_{\text{enc}}| A_j'^{\dagger}. \tag{3}$$

The noise operators A_j' act on the codeword $|\psi_{\text{enc}}\rangle$ which lives in a larger dimensional space than that of the original quantum state $|\psi\rangle$. The noise operators A_j act on $|\psi\rangle$. Notice that in general,

$$\text{Tr}_{\text{anc}}\left[\sum_k U_{\text{enc}}^{\dagger}\left(\sum_j A_j' U_{\text{enc}} |\psi\rangle |00...0\rangle \langle 00...0| \langle\psi| U_{\text{enc}}^{\dagger} A_j'^{\dagger}\right) U_{\text{enc}}\right] \neq |\psi\rangle\langle\psi|\,. \tag{4}$$

Therefore, if we encode the quantum information $|\psi\rangle$, subject it to the noise A_j' and decode using the inverse of the encoding operation, $U_{\text{enc}}^{\dagger} = U_{\text{enc}}^{-1} \overset{\text{def}}{=} U_{\text{dec}}$, we will not always recover the original state $|\psi\rangle$. To recover $|\psi\rangle$ we need to

introduce an error recovery operation $\mathcal{R}$ that has the effect of undoing enough of the noise A'_j on the codeword state $|\psi_{\text{enc}}\rangle$ so that after decoding and tracing out we are left with $|\psi\rangle$ [2],

$$\mathrm{Tr}_{\text{anc}}\left[\sum_k \mathcal{R}_k U_{\text{enc}}^{\dagger}\left(\sum_j A'_j U_{\text{enc}}\,|\psi\rangle\,|00...0\rangle\,\langle 00...0|\,\langle\psi|\,U_{\text{enc}}^{\dagger} A_j'^{\dagger}\right) U_{\text{enc}} \mathcal{R}_k^{\dagger}\right] = |\psi\rangle\,\langle\psi|\,. \tag{5}$$

The design of a quantum error correcting code can be reduced to finding a unitary encoding operator U_{enc} and a recovery operation $\mathcal{R}$ so that, given an error model corresponding to a specified set of error operators A_j, equation (5) is valid. The action of the recovery operator $\mathcal{R}$ may be interpreted as pushing all the noise into the ancilla $|00...0\rangle$ so that the errors are eliminated when the ancilla is traced out.

Computing the Entanglement Fidelity. In [14], Schumacher introduced the concept of entanglement fidelity as a useful indicator of the efficiency of quantum error correcting codes. The entanglement fidelity is defined for a mixed state $\rho = \sum_i p_i \rho_i = \mathrm{tr}_{\mathcal{H}_R}|\psi\rangle\,\langle\psi|$ in terms of a purification $|\psi\rangle \in \mathcal{H}\otimes\mathcal{H}_R$ to a reference system $\mathcal{H}_R$. The purification $|\psi\rangle$ encodes all of the information in ρ. Entanglement fidelity is a measure of how well the channel Λ preserves the entanglement of the state $\mathcal{H}$ with its reference system $\mathcal{H}_R$. The entanglement fidelity is defined as follows [14],

$$\mathcal{F}(\rho,\,\Lambda) \stackrel{\text{def}}{=} \langle\psi|\,(\Lambda\otimes I_{\mathcal{H}_R})\,(|\psi\rangle\,\langle\psi|)\,|\psi\rangle\,, \tag{6}$$

where $|\psi\rangle$ is any purification of ρ, $I_{\mathcal{H}_R}$ is the identity map on $\mathcal{M}(\mathcal{H}_R)$ and $\Lambda\otimes I_{\mathcal{H}_R}$ is the evolution operator extended to the space $\mathcal{H}\otimes\mathcal{H}_R$, space on which ρ has been purified. If the quantum operation Λ is written in terms of its Kraus operator elements $\{A_k\}$ as, $\Lambda(\rho) = \sum_k A_k \rho A_k^{\dagger}$, then it can be shown that [16],

$$\mathcal{F}(\rho,\,\Lambda) = \sum_k \mathrm{tr}\,(A_k\rho)\,\mathrm{tr}\left(A_k^{\dagger}\rho\right) = \sum_k |\mathrm{tr}\,(\rho A_k)|^2\,. \tag{7}$$

This expression for the entanglement fidelity is very useful for explicit calculations. Finally, assuming that

$$\Lambda : \mathcal{M}(\mathcal{H}) \ni \rho \longmapsto \Lambda(\rho) = \sum_k A_k \rho A_k^{\dagger} \in \mathcal{M}(\mathcal{H})\,,\ \dim_{\mathbb{C}}\mathcal{H} = N \tag{8}$$

and choosing a purification described by a maximally entangled unit vector $|\psi\rangle \in \mathcal{H}\otimes\mathcal{H}$ for the mixed state $\rho = \frac{1}{\dim_{\mathbb{C}}\mathcal{H}} I_{\mathcal{H}}$, we obtain

$$\mathcal{F}\left(\frac{1}{N} I_{\mathcal{H}},\,\Lambda\right) = \frac{1}{N^2}\sum_k |\mathrm{tr} A_k|^2\,. \tag{9}$$

The expression in (9) represents the entanglement fidelity when no error correction is performed on the noisy channel Λ in (8). In this Letter, we will follow the general theoretical framework describing requirements for quantum error correcting codes presented in [4].

B. Error Models

To introduce noise models, we assume a quantum channel (CPTP map) Λ on n-uses to be expressible by means of the following Kraus decomposition,

$$\Lambda^{(n)}(\rho) = \sum_{i_1,\ldots,\,i_n} p\,(i_n,\,i_{n-1},\ldots,\,i_1)\,(A_{i_n}\otimes\ldots\otimes A_{i_1})\,\rho\,(A_{i_n}\otimes\ldots\otimes A_{i_1})^{\dagger}\,. \tag{10}$$

If the probability $p\,(i_n,\,i_{n-1},\ldots,\,i_1)$ is factorized in the product of n-independent probabilities, $p\,(i_n,\,i_{n-1},\ldots,\,i_1) = \prod_{l=1}^{n} p\,(i_l)$, we are in the presence of a memoryless channel and $\Lambda^{(n)}(\rho) = \Lambda^{\otimes n}(\rho) \stackrel{\text{def}}{=} \Lambda\otimes\ldots\otimes\Lambda$. On the contrary if $p\,(i_n,\,i_{n-1},\ldots,\,i_1)$ is not separable in the product of of n-independent probabilities, then Λ is a memory channel with $\Lambda^{(n)}(\rho) \neq \Lambda^{\otimes n}(\rho)$. For instance, a very important class of quantum memory channels is described by the Markovian correlated noise channels of length n,

$$\Lambda^{(n)}(\rho) \stackrel{\text{def}}{=} \sum_{i_1,\ldots,\,i_n} p\,(i_n|i_{n-1})\,p\,(i_{n-1}|i_{n-2})\ldots p\,(i_2|i_1)\,p_{i_1}\,(A_{i_n}\otimes\ldots\otimes A_{i_1})\,\rho\,(A_{i_n}\otimes\ldots\otimes A_{i_1})^{\dagger}\,, \tag{11}$$

with $p\,(i_l|i_{l-1}) \stackrel{\text{def}}{=} (1-\mu)\,p_{i_l} + \mu\delta_{i_l,\,i_{l-1}}$, $\forall l = 1,\ldots,\,n$. The correlation parameter μ describes the degree of memory of the channel considered.

III. REPETITION CODES FOR CORRELATED BIT FLIP

In this Section, we introduce a simple error model in the presence of correlated errors. Specifically, we consider bit flip (or phase flip) noisy quantum memory channels and QEC is performed via odd and even repetition codes (RC) [17]. Although the error models considered are classical in nature, from this preliminary work we hope to gain useful insights for extending error correction techniques to quantum correlated error models. The performance of quantum error correcting codes is quantified by means of the entanglement fidelity $\mathcal{F}_{RC}^{(n)}(\mu,\, p)$ as function of the error probability p and degree of memory μ.

A. CASE, $n_{\mathbf{odd}}=3$

Consider n qubits and correlated errors in a bit flip quantum channel,

$$\Lambda^{(n)}(\rho)\overset{\text{def}}{=}\sum_{i_1,\ldots,\, i_n=0}^{1} p_{i_n|i_{n-1}}p_{i_{n-1}|i_{n-2}}\ldots p_{i_2|i_1}p_{i_1}\left(A_{i_n}\otimes\ldots\otimes A_{i_1}\right)\rho\left(A_{i_n}\otimes\ldots\otimes A_{i_1}\right)^{\dagger}, \tag{12}$$

where $A_0\overset{\text{def}}{=}I$, $A_1\overset{\text{def}}{=}X$ are Pauli operators. Furthermore,

$$p_{i_k|i_j}=(1-\mu)p_{i_k}+\mu\delta_{i_k,\, i_j},\quad p_{i_k=0}=1-p,\, p_{i_k=1}=p, \tag{13}$$

with,

$$\sum_{i_1,\ldots,\, i_n=0}^{1} p_{i_n|i_{n-1}}p_{i_{n-1}|i_{n-2}}\ldots p_{i_2|i_1}p_{i_1}=1. \tag{14}$$

To simplify our notation, we may assume that $A_{i_n}\otimes\ldots\otimes A_{i_1}\equiv A_{i_n}\ldots A_{i_1}$.

Error Operators. In the simplest example, consider three qubits ($n=3$) and correlated errors in a bit flip channel,

$$\Lambda^{(3)}(\rho)\overset{\text{def}}{=}\sum_{i_1,\, i_2,\, i_3=0}^{1} p_{i_3|i_2}p_{i_2|i_1}p_{i_1}\left[A_{i_3}A_{i_2}A_{i_1}\rho A_{i_1}^{\dagger}A_{i_2}^{\dagger}A_{i_3}^{\dagger}\right],\text{ with }\sum_{i_1,\, i_2,\, i_3=0}^{1} p_{i_3|i_2}p_{i_2|i_1}p_{i_1}=1. \tag{15}$$

Substituting (13) in (15), it follows that the error superoperator $\mathcal{A}$ associated to channel (15) is defined in terms of the following error operators,

$$\mathcal{A}\longleftrightarrow\{A_0',\ldots,\, A_7'\}\text{ with }\Lambda^{(3)}(\rho)\overset{\text{def}}{=}\sum_{k=0}^{7}A_k'\rho A_k'^{\dagger}\text{ and, }\sum_{k=0}^{7}A_k'^{\dagger}A_k'=I_{8\times 8}. \tag{16}$$

In an explicit way, the error operators $\{A_0',\ldots,\, A_7'\}$ are given by,

$$\begin{aligned}
A_0' &= \sqrt{\tilde{p}_0^{(3)}}I^1\otimes I^2\otimes I^3,\ A_1'=\sqrt{\tilde{p}_1^{(3)}}X^1\otimes I^2\otimes I^3,\ A_2'=\sqrt{\tilde{p}_2^{(3)}}I^1\otimes X^2\otimes I^3,\\
A_3' &= \sqrt{\tilde{p}_3^{(3)}}I^1\otimes I^2\otimes X^3,\ A_4'=\sqrt{\tilde{p}_4^{(3)}}X^1\otimes X^2\otimes I^3,\ A_5'=\sqrt{\tilde{p}_5^{(3)}}X^1\otimes I^2\otimes X^3,\\
A_6' &= \sqrt{\tilde{p}_6^{(3)}}I^1\otimes X^2\otimes X^3,\ A_7'=\sqrt{\tilde{p}_7^{(3)}}X^1\otimes X^2\otimes X^3,
\end{aligned} \tag{17}$$

where the coefficients $\tilde{p}_k^{(3)}$ for $k=1,..,\,7$ are given by,

$$\begin{aligned}
\tilde{p}_0^{(3)} &= p_{00}^2p_0,\ \tilde{p}_1^{(3)}=p_{00}p_{10}p_0,\ \tilde{p}_2^{(3)}=p_{01}p_{10}p_0,\ \tilde{p}_3^{(3)}=p_{00}p_{01}p_1,\\
\tilde{p}_4^{(3)} &= p_{10}p_{11}p_0,\ \tilde{p}_5^{(3)}=p_{01}p_{10}p_1,\ \tilde{p}_6^{(3)}=p_{01}p_{11}p_1,\ \tilde{p}_7^{(3)}=p_{11}^2p_1,
\end{aligned} \tag{18}$$

with,

$$p_0 = (1-p), p_1 = p, p_{00} = ((1-\mu)(1-p)+\mu),$$

$$p_{01} = (1-\mu)(1-p), p_{10} = (1-\mu)p, p_{11} = ((1-\mu)p+\mu). \tag{19}$$

Encoding and Decoding Operators. Consider a repetition code that encodes 1 logical qubit into 3-physical qubits. We have,

$$|0\rangle \overset{\text{Tensoring}}{\longrightarrow} |0\rangle \otimes |00\rangle = |000\rangle \overset{\text{def}}{=} |0_{\text{L}}\rangle, \ |1\rangle \overset{\text{tensoring}}{\longrightarrow} |1\rangle \otimes |00\rangle = |100\rangle \overset{U_{\text{CNOT}}^{12}\otimes I^3}{\longrightarrow} |110\rangle \overset{U_{\text{CNOT}}^{13}\otimes I^2}{\longrightarrow} |111\rangle \overset{\text{def}}{=} |1_{\text{L}}\rangle. \tag{20}$$

The operator U_{CNOT}^{ij} is the CNOT gate from qubit i to j defined as,

$$U_{\text{CNOT}}^{ij} \overset{\text{def}}{=} \frac{1}{2}\left[\left(I^i + Z^i\right)\otimes I^j + \left(I^i - Z^i\right)\otimes X^j\right]. \tag{21}$$

Finally, the encoding operator U_{enc} such that $U_{\text{enc}}|000\rangle = |000\rangle$ and $U_{\text{enc}}|100\rangle = |111\rangle$ is defined as,

$$U_{\text{enc}} \overset{\text{def}}{=} \left(U_{\text{CNOT}}^{13}\otimes I^2\right)\circ\left(U_{\text{CNOT}}^{12}\otimes I^3\right). \tag{22}$$

Recovery Operators. The set of error operators satisfying the detectability condition [17], $P_{\mathcal{C}} A_k' P_{\mathcal{C}} = \lambda_{A_k'} P_{\mathcal{C}}$, where $P_{\mathcal{C}} = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projector operator on the code subspace $\mathcal{C} = Span\{|0_L\rangle, |1_L\rangle\}$ is given by,

$$\mathcal{A}_{\text{detectable}} = \{A_0', A_1', A_2', A_3', A_4', A_5', A_6'\} \subseteq \mathcal{A}. \tag{23}$$

The only non-detectable error is A_7'. Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $\mathcal{A}_{\text{correctable}}^\dagger \mathcal{A}_{\text{correctable}}$ is detectable. It follows then that,

$$\mathcal{A}_{\text{correctable}} = \{A_0', A_1', A_2', A_3'\} \subseteq \mathcal{A}_{\text{detectable}} \subseteq \mathcal{A}. \tag{24}$$

The action of the correctable error operators $\mathcal{A}_{\text{correctable}}$ on the codewords $|0_L\rangle$ and $|1_L\rangle$ is given by,

$$|0_L\rangle \rightarrow A_0'|0_L\rangle = \sqrt{\tilde{p}_0^{(3)}}|000\rangle, A_1'|0_L\rangle = \sqrt{\tilde{p}_1^{(3)}}|100\rangle, A_2'|0_L\rangle = \sqrt{\tilde{p}_2^{(3)}}|010\rangle, A_3'|0_L\rangle = \sqrt{\tilde{p}_3^{(3)}}|001\rangle$$

$$|1_L\rangle \rightarrow A_0'|1_L\rangle = \sqrt{\tilde{p}_0^{(3)}}|111\rangle, A_1'|1_L\rangle = \sqrt{\tilde{p}_1^{(3)}}|011\rangle, A_2'|1_L\rangle = \sqrt{\tilde{p}_2^{(3)}}|101\rangle, A_3'|1_L\rangle = \sqrt{\tilde{p}_3^{(3)}}|110\rangle. \tag{25}$$

The two four-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_2^3$ generated by the action of $\mathcal{A}_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$\mathcal{V}^{0_L} = Span\left\{\left|v_1^{0_L}\right\rangle = |000\rangle, \left|v_2^{0_L}\right\rangle = |100\rangle, \left|v_3^{0_L}\right\rangle = |010\rangle, \left|v_4^{0_L}\right\rangle = |001\rangle\right\}, \tag{26}$$

and,

$$\mathcal{V}^{1_L} = Span\left\{\left|v_1^{1_L}\right\rangle = |111\rangle, \left|v_2^{1_L}\right\rangle = |011\rangle, \left|v_3^{1_L}\right\rangle = |101\rangle, \left|v_4^{1_L}\right\rangle = |110\rangle\right\}, \tag{27}$$

respectively. Notice that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} = \mathcal{H}_2^3$. The recovery superoperator $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1,..,4$ is defined as [4],

$$R_l \overset{\text{def}}{=} V_l \sum_{i=0}^{1} \left|v_l^{i_L}\right\rangle\left\langle v_l^{i_L}\right|, \tag{28}$$

where the unitary operator V_l is such that $V_l\left|v_l^{i_L}\right\rangle = |i_L\rangle$ for $i \in \{0, 1\}$. Substituting (26) and (27) into (28), it follows that the four recovery operators $\{R_1, R_2, R_3, R_4\}$ are given by,

$$R_1 = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|, R_2 = |0_L\rangle\langle 100| + |1_L\rangle\langle 011|,$$

$$R_3 = |0_L\rangle\langle 010| + |1_L\rangle\langle 101|, R_4 = |0_L\rangle\langle 001| + |1_L\rangle\langle 110|. \tag{29}$$

Using simple algebra, it turns out that the 8×8 matrix representation $[R_l]$ with $l=1,..,4$ of the recovery operators is given by,

$$[R_1]=E_{11}+E_{88},\ [R_2]=E_{12}+E_{87},\ [R_3]=E_{13}+E_{86},\ [R_4]=E_{14}+E_{85}, \tag{30}$$

where E_{ij} is the 8×8 matrix where the only non-vanishing element is the one located in the ij-position and it equals 1. It follows that $\mathcal{R}\leftrightarrow\{R_l\}$ is indeed a trace preserving quantum operation since,

$$\sum_{l=1}^{4} R_l^{\dagger}R_l=I_{8\times 8}. \tag{31}$$

Considering this recovery operation $\mathcal{R}$, the map $\Lambda^{(3)}(\rho)$ in (16) becomes,

$$\Lambda^{(3)}_{\text{recover}}(\rho)\equiv\left(\mathcal{R}\circ\Lambda^{(3)}\right)(\rho)\overset{\text{def}}{=}\sum_{k=0}^{7}\sum_{l=1}^{4}\left(R_lA_k'\right)\rho\left(R_lA_k'\right)^{\dagger}. \tag{32}$$

Entanglement Fidelity. We want to describe the action of $\mathcal{R}\circ\Lambda^{(3)}$ restricted to the code subspace $\mathcal{C}$. Therefore, we compute the 2×2 matrix representation $[R_lA_k']_{|\mathcal{C}}$ of each R_lA_k' with $l=1,..,4$ and $k=0,..,7$ where,

$$[R_lA_k']_{|\mathcal{C}}\overset{\text{def}}{=}\begin{pmatrix}\langle 0_L|R_lA_k'|0_L\rangle & \langle 0_L|R_lA_k'|1_L\rangle\\ \langle 1_L|R_lA_k'|0_L\rangle & \langle 1_L|R_lA_k'|1_L\rangle\end{pmatrix}. \tag{33}$$

Substituting (25) and (29) into (33), it turns out that the only matrices $[R_lA_k']_{|\mathcal{C}}$ with non-vanishing trace are given by,

$$\begin{aligned}[][R_1A_0']_{|\mathcal{C}} &= \sqrt{\tilde{p}_0^{(3)}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\ [R_2A_1']_{|\mathcal{C}}=\sqrt{\tilde{p}_1^{(3)}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\\ [R_3A_2']_{|\mathcal{C}} &= \sqrt{\tilde{p}_2^{(3)}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\ [R_4A_3']_{|\mathcal{C}}=\sqrt{\tilde{p}_3^{(3)}}\begin{pmatrix}1&0\\0&1\end{pmatrix}.\end{aligned} \tag{34}$$

Therefore, the entanglement fidelity $\mathcal{F}^{(3)}_{RC}(\mu,\,p)$ defined as,

$$\mathcal{F}^{(3)}_{RC}(\mu,\,p)\overset{\text{def}}{=}\mathcal{F}^{(3)}\left(\frac{1}{2}I_{2\times 2},\,\mathcal{R}\circ\Lambda^{(3)}\right)=\frac{1}{(2)^2}\sum_{k=0}^{7}\sum_{l=1}^{4}\left|\text{tr}\left([R_lA_k']_{|\mathcal{C}}\right)\right|^2, \tag{35}$$

results,

$$\mathcal{F}^{(3)}_{RC}(\mu,\,p)=\tilde{p}_0^{(3)}+\tilde{p}_1^{(3)}+\tilde{p}_2^{(3)}+\tilde{p}_3^{(3)}. \tag{36}$$

The expression for $\mathcal{F}^{(3)}_{RC}(\mu,\,p)$ in (35) represents the entanglement fidelity quantifying the performance of the error correction scheme provided by the repetition code here considered. The quantum operation $\mathcal{R}\circ\Lambda^{(3)}$ appearing in (35) is defined in equation (32) and the recovery operators R_l are explicitly given in (29). The action of R_lA_k' in (35) is restricted to the code space $\mathcal{C}$ defined in (20).

Substituting (18) and (19) into (36), we finally obtain

$$\mathcal{F}^{(3)}_{RC}(\mu,\,p)=\mu^2\left(2p^3-3p^2+p\right)+\mu\left(-4p^3+6p^2-2p\right)+\left(2p^3-3p^2+1\right). \tag{37}$$

Notice that for a vanishing degree of memory μ, the entanglement fidelity becomes,

$$\mathcal{F}^{(3)}_{RC}(0,\,p)=\sum_{m=0}^{1}\binom{3}{m}p^m(1-p)^{3-m}=2p^3-3p^2+1. \tag{38}$$

Remarks on the coding for phase flip memory channels. The code for the phase flip channel has the same characteristics as the code for the bit flip channel. These two channels are unitarily equivalent since there is a unitary operator, the Hadamard gate H, such that the action of one channel is the same as the other, provided the first channel is preceded by H and followed by $H^{\dagger}$ [1],

$$\Lambda^{\text{phase}}(\rho)\overset{\text{def}}{=}\left(H\circ\Lambda^{\text{bit}}\circ H^{\dagger}\right)(\rho)=(1-p)\rho+pZ\rho Z, \tag{39}$$

where,

$$\Lambda^{\text{bit}}(\rho) \overset{\text{def}}{=} (1-p)\rho + pX\rho X \text{ and, } H \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{40}$$

These operations may be trivially incorporated into the encoding and error-correction operations. The encoding for the phase flip channel is performed in two steps: i) first, we encode in three qubits exactly as for the bit flip channel; ii) second, we apply a Hadamard gate to each qubit,

$$|0\rangle \xrightarrow{\text{tensor}} |000\rangle \xrightarrow{U_{\text{enc}}^{\text{bit}}} |000\rangle \xrightarrow{H^{\otimes 3}} |0_{\text{L}}\rangle \overset{\text{def}}{=} |+++\rangle, \; |1\rangle \xrightarrow{\text{tensor}} |100\rangle \xrightarrow{U_{\text{enc}}^{\text{bit}}} |111\rangle \xrightarrow{H^{\otimes 3}} |1_{\text{L}}\rangle \overset{\text{def}}{=} |---\rangle, \tag{41}$$

where,

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}. \tag{42}$$

The unitary encoding operator $U_{\text{enc}}^{\text{phase}}$ is given by $U_{\text{enc}}^{\text{phase}} \overset{\text{def}}{=} H^{\otimes 3} \circ U_{\text{enc}}^{\text{bit}}$,with $U_{\text{enc}}^{\text{bit}}$ defined in (22). Furthermore, in the phase flip code, the recovery operation is the Hadamard conjugated recovery operation from the bit flip code, $R_k^{\text{phase}} \overset{\text{def}}{=} H^{\otimes 3} R_k^{\text{bit}} H^{\otimes 3}$.

B. CASE, $n_{\text{even}} = 4$

Here, we apply the even length repetition code with $n = 4$ to the correlated bit-flip.

Error Operators. In this case, the memory channel to consider is given by,

$$\Lambda^{(4)}(\rho) \overset{\text{def}}{=} \sum_{i_1,\, i_2,\, i_3,\, i_4=0}^{1} p_{i_4|i_3} p_{i_3|i_2} p_{i_2|i_1} p_{i_1} \left[A_{i_4} A_{i_3} A_{i_2} A_{i_1} \rho A_{i_1}^{\dagger} A_{i_2}^{\dagger} A_{i_3}^{\dagger} A_{i_4}^{\dagger} \right], \tag{43}$$

where the error operators $\{A_{i_r}\}$ with $r = 1, 2, 3, 4$ act on 1 qubit quantum states. The error superoperator $\mathcal{A}$ associated to channel (43) may be defined in terms of the following encoded error operators $\{A_k'\}$ with $k = 0,..,15$,

$$\mathcal{A} \longleftrightarrow \{A_0',.., A_{15}'\} \text{ with } \Lambda^{(4)}(\rho) \overset{\text{def}}{=} \sum_{k=0}^{15} A_k' \rho A_k'^{\dagger} \text{ and, } \sum_{k=0}^{15} A_k'^{\dagger} A_k' = I_{16\times 16}. \tag{44}$$

Omitting the symbol "$\otimes$", the error operators $\{A_0',.., A_{15}'\}$ are given by,

$$\begin{aligned}
A_0' &= \sqrt{\tilde{p}_0^{(4)}} I^1 I^2 I^3 I^4,\; A_1' = \sqrt{\tilde{p}_1^{(4)}} X^1 I^2 I^3 I^4,\; A_2' = \sqrt{\tilde{p}_2^{(4)}} I^1 X^2 I^3 I^4,\; A_3' = \sqrt{\tilde{p}_3^{(4)}} I^1 I^2 X^3 I^4,\; A_4' = \sqrt{\tilde{p}_4^{(4)}} I^1 I^2 I^3 X^4, \\
A_5' &= \sqrt{\tilde{p}_5^{(4)}} X^1 X^2 I^3 I^4,\; A_6' = \sqrt{\tilde{p}_6^{(4)}} X^1 I^2 X^3 I^4,\; A_7' = \sqrt{\tilde{p}_7^{(4)}} X^1 I^2 I^3 X^4,\; A_8' = \sqrt{\tilde{p}_8^{(4)}} I^1 X^2 X^3 I^4,\; A_9' = \sqrt{\tilde{p}_9^{(4)}} I^1 X^2 I^3 X^4, \\
A_{10}' &= \sqrt{\tilde{p}_{10}^{(4)}} I^1 I^2 X^3 X^4,\; A_{11}' = \sqrt{\tilde{p}_{11}^{(4)}} X^1 X^2 X^3 I^4,\; A_{12}' = \sqrt{\tilde{p}_{12}^{(4)}} X^1 X^2 I^3 X^4,\; A_{13}' = \sqrt{\tilde{p}_{13}^{(4)}} X^1 I^2 X^3 X^4 \\
A_{14}' &= \sqrt{\tilde{p}_{14}^{(4)}} I^1 X^2 X^3 X^4,\; A_{15}' = \sqrt{\tilde{p}_{15}^{(4)}} X^1 X^2 X^3 X^4.
\end{aligned} \tag{45}$$

The coefficients $\tilde{p}_k^{(4)}$ for $k = 1,.., 15$ are formally given by $\tilde{p}_k^{(4)} = p_{i_1^{(4)} i_2^{(4)}} p_{i_2^{(4)} i_3^{(4)}} p_{i_3^{(4)} i_4^{(4)}} p_{i_4^{(4)}}$ where the $i_j^{(4)} \in \{0, 1\}$ are determined by the relation $A_k' |0_L\rangle = \left| i_1^{(4)} i_2^{(4)} i_3^{(4)} i_4^{(4)} \right\rangle$. Following the line of reasoning used for the odd repetition codes and omitting technical details that will appear in Appendix A, the entanglement fidelity $\mathcal{F}_{RC}^{(4)}(\mu, p)$ becomes,

$$\mathcal{F}_{RC}^{(4)}(\mu, p) = \mu^2 \left(2p^3 - 3p^2 + p\right) + \mu\left(-4p^3 + 6p^2 - 2p\right) + \left(2p^3 - 3p^2 + 1\right). \tag{46}$$

Notice that $\mathcal{F}_{RC}^{(4)}(\mu, p) = \mathcal{F}_{RC}^{(3)}(\mu, p)$ and, in absence of correlations,

$$\mathcal{F}_{RC}^{(4)}(0, p) = \sum_{m=0}^{1} \binom{4}{m} p^m (1-p)^{4-m} + \frac{1}{2}\binom{4}{2} p^2 (1-p)^2 = 2p^3 - 3p^2 + 1 \equiv \mathcal{F}_{RC}^{(3)}(0, p). \tag{47}$$

Finally, it can also be shown that $\mathcal{F}_{RC}^{(6)}(\mu,\, p) = \mathcal{F}_{RC}^{(5)}(\mu,\, p)$ with,

$$\begin{aligned}\mathcal{F}_{RC}^{(5)}(\mu,\, p) \;=\;& \mu^4\left(-6p^5+15p^4-12p^3+3p^2\right)+\mu^3\left(24p^5-60p^4+52p^3-18p^2+2p\right)+\\ &+\mu^2\left(-36p^5+90p^4-78p^3+27p^2-3p\right)+\mu\left(24p^5-60p^4+48p^3-12p^2\right)+\\ &+\left(-6p^5+15p^4-10p^3+1\right), \end{aligned} \tag{48}$$

and $\mathcal{F}_{RC}^{(8)}(\mu,\, p) = \mathcal{F}_{RC}^{(7)}(\mu,\, p)$ with,

$$\begin{aligned}\mathcal{F}_{RC}^{(7)}(\mu,\, p) \;=\;& \mu^6\left(20p^7-70p^6+90p^5-50p^4+10p^3\right)+\mu^5\left(-120p^7+420p^6-564p^5+360p^4-108p^3+12p^2\right)+\\ &+\mu^4\left(300p^7-1050p^6+1440p^5-975p^4+336p^3-54p^2+3p\right)+\\ &+\mu^3\left(-400p^7+1400p^6-1920p^5+1300p^4-448p^3+72p^2-4p\right)+\\ &+\mu^2\left(300p^7-1050p^6+1410p^5-900p^4+270p^3-30p^2\right)+\mu\left(-120p^7+420p^6-540p^5+300p^4-60p^3\right)+\\ &+\left(20p^7-70p^6+84p^5-35p^4+1\right). \end{aligned} \tag{49}$$

In Figure 1, we plot $\mathcal{F}_{RC}^{(3)}(\mu,\, p)$, $\mathcal{F}_{RC}^{(5)}(\mu,\, p)$ and $\mathcal{F}_{RC}^{(7)}(\mu,\, p)$ vs. μ for $p = 0.45$. From this plot, it is clear that the entanglement fidelity $\mathcal{F}_{RC}^{(n)}(\mu,\, p)$ increases with increasing n and decreases with the correlation parameter μ.

IV. DECOHERENCE FREE SUBSPACES FOR CORRELATED BIT FLIP

In this Section, we tackle our decoherence problem via the decoherence-free subspaces formalism. This is a passive quantum error correction method where the key idea is that of avoiding decoherence by encoding quantum information into special subspaces that are protected from the interaction with the environment by virtue of some specific dynamical symmetry. For a detailed review, we refer to [15].

A. CASE, $n_{\text{odd}} = 3$

Let us consider the correlated bit-flip noisy error model as defined in (15) and (16).

Encoding and Decoding Operators. Consider the following quantum code encoding 1 logical qubit into 3-physical qubits,

$$\begin{aligned}|0\rangle \;\rightarrow\; |0_L\rangle &\overset{\text{def}}{=} \frac{1}{\left(\sqrt{2}\right)^3}\left(|0\rangle_1+|1\rangle_1\right)\otimes\left(|0\rangle_2+|1\rangle_2\right)\otimes\left(|0\rangle_3+|1\rangle_3\right)\equiv|+++\rangle,\\ |1\rangle \;\rightarrow\; |1_L\rangle &\overset{\text{def}}{=} \frac{1}{\left(\sqrt{2}\right)^3}\left(|0\rangle_1-|1\rangle_1\right)\otimes\left(|0\rangle_2-|1\rangle_2\right)\otimes\left(|0\rangle_3-|1\rangle_3\right)\equiv|---\rangle, \end{aligned} \tag{50}$$

with $\langle +++|+++\rangle = \langle ---|---\rangle = 1$ and $\langle ---|+++\rangle = \langle +++|---\rangle = 0$.

Recovery Operators. The set of error operators satisfying the detectability condition $P_{\mathcal{C}}A_k^{\prime}P_{\mathcal{C}} = \lambda_{A_k^{\prime}}P_{\mathcal{C}}$ where $P_{\mathcal{C}} = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projector operator on the code subspace $\mathcal{C} = Span\{|0_L\rangle,\, |1_L\rangle\}$ is given by,

$$\mathcal{A}_{\text{detectable}} = \{A_0^{\prime},\, A_4^{\prime},\, A_5^{\prime},\, A_6^{\prime}\} \subseteq \mathcal{A}. \tag{51}$$

Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $\mathcal{A}_{\text{correctable}}^{\dagger}\mathcal{A}_{\text{correctable}}$ is detectable. It follows then that,

$$\mathcal{A}_{\text{correctable}} = \{A_0^{\prime},\, A_4^{\prime},\, A_5^{\prime},\, A_6^{\prime}\} \equiv \mathcal{A}_{\text{detectable}}. \tag{52}$$

The action of the correctable error operators $\mathcal{A}_{\text{correctable}}$ on the codewords $|0_L\rangle$ and $|1_L\rangle$ is given by,

$$|0_L\rangle \to A'_r|0_L\rangle = \sqrt{\tilde{p}_r^{(3)}}|0_L\rangle\,,\ A'_r|1_L\rangle = \sqrt{\tilde{p}_r^{(3)}}|1_L\rangle\,, \tag{53}$$

for $r = 0, 4, 5, 6$. From (53), it follows that $\mathcal{C} = Span\{|0_L\rangle\,,\ |1_L\rangle\}$ is a decoherence-free subspace for the correctable error operators in $\mathcal{A}_{\text{correctable}}$. The two one-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_2^3$ generated by the action of $\mathcal{A}_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$\mathcal{V}^{0_L} = Span\left\{\left|v_1^{0_L}\right\rangle = |+++\rangle\right\}, \tag{54}$$

and,

$$\mathcal{V}^{1_L} = Span\left\{\left|v_1^{1_L}\right\rangle = |---\rangle\right\}, \tag{55}$$

respectively. Notice that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} \neq \mathcal{H}_2^3$. This means that the trace preserving recovery superoperator $\mathcal{R}$ is defined in terms of one standard recovery operator R_1 and by the projector $R_\perp$ onto the orthogonal complement of $\bigoplus_{i=0}^{1}\mathcal{V}^{i_L}$, i. e. the part of the Hilbert space $\mathcal{H}_2^3$ which is not reached by acting on the code $\mathcal{C}$ with the correctable error operators. In the case under consideration,

$$R_1 \overset{\text{def}}{=} |+++\rangle\langle+++| + |---\rangle\langle---|\,,\ R_\perp = \sum_{s=1}^{6}|r_s\rangle\langle r_s|\,, \tag{56}$$

where $\{|r_s\rangle\}$ is an orthonormal basis for $\left(\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L}\right)^\perp$. A suitable basis $\mathcal{B}_{\left(\mathcal{V}^{0_L}\oplus\mathcal{V}^{1_L}\right)^\perp}$ is given by,

$$\mathcal{B}_{\left(\mathcal{V}^{0_L}\oplus\mathcal{V}^{1_L}\right)^\perp} = \{r_1 = |-++\rangle\,,\ r_2 = |+-+\rangle\,,\ r_3 = |++-\rangle\,,\ r_4 = |--+\rangle\,,\ r_5 = |-+-\rangle\,,\ r_6 = |+--\rangle\}\,. \tag{57}$$

Therefore, $\mathcal{R} \leftrightarrow \{R_1,\, R_\perp\}$ is indeed a trace preserving quantum operation,

$$R_1^\dagger R_1 + R_\perp^\dagger R_\perp = I_{8\times 8}. \tag{58}$$

Considering this recovery operation $\mathcal{R}$ with $R_2 \equiv R_\perp$, the map $\Lambda^{(3)}(\rho)$ in (16) becomes,

$$\Lambda_{\text{recover}}^{(3)}(\rho) \equiv \left(\mathcal{R}\circ\Lambda^{(3)}\right)(\rho) \overset{\text{def}}{=} \sum_{k=0}^{7}\sum_{l=1}^{2}\left(R_l A'_k\right)\rho\left(R_l A'_k\right)^\dagger\,, \tag{59}$$

Entanglement Fidelity. We want to describe the action of $\mathcal{R}\circ\Lambda^{(3)}$ restricted to the code subspace $\mathcal{C}$. Therefore, we compute the 2×2 matrix representation $[R_l A'_k]_{|\mathcal{C}}$ of each $R_l A'_k$ with $l = 1, 2$ and $k = 0, .., 7$ where,

$$[R_l A'_k]_{|\mathcal{C}} \overset{\text{def}}{=} \begin{pmatrix} \langle 0_L|R_l A'_k|0_L\rangle & \langle 0_L|R_l A'_k|1_L\rangle \\ \langle 1_L|R_l A'_k|0_L\rangle & \langle 1_L|R_l A'_k|1_L\rangle \end{pmatrix}. \tag{60}$$

Substituting (53) and (56) into (60), it turns out that the only matrices $[R_l A'_k]_{|\mathcal{C}}$ with non-vanishing trace are given by,

$$[R_1 A'_r]_{|\mathcal{C}} = \sqrt{\tilde{p}_r^{(3)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{61}$$

for $r = 0, 4, 5, 6$. Therefore, the entanglement fidelity $\mathcal{F}_{DFS}^{(3)}(\mu,\, p)$ defined as,

$$\mathcal{F}_{DFS}^{(3)}(\mu,\, p) \overset{\text{def}}{=} \mathcal{F}^{(3)}\left(\frac{1}{2}I_{2\times 2},\, \mathcal{R}\circ\Lambda^{(3)}\right) = \frac{1}{(2)^2}\sum_{k=0}^{7}\sum_{l=1}^{2}\left|\text{tr}\left([R_l A'_k]_{|\mathcal{C}}\right)\right|^2, \tag{62}$$

results,

$$\mathcal{F}_{DFS}^{(3)}(\mu,\, p) = \tilde{p}_0^{(3)} + \tilde{p}_4^{(3)} + \tilde{p}_5^{(3)} + \tilde{p}_6^{(3)}. \tag{63}$$

The expression for $\mathcal{F}_{DFS}^{(3)}(\mu,\, p)$ in (62) represents the entanglement fidelity quantifying the performance of the error correction scheme provided by the noiseless code here considered. The quantum operation $\mathcal{R}\circ\Lambda^{(3)}$ appearing in (62) is defined in equation (59) and the recovery operators R_l are explicitly given in (56). The action of $R_l A'_k$ in (62) is restricted to the code space $\mathcal{C}$ defined in (50).

Substituting (19) into (63), we finally obtain

$$\mathcal{F}_{DFS}^{(3)}(\mu,\, p) = \mu^2\left(-4p^3 + 6p^2 - 2p\right) + \mu\left(8p^3 - 12p^2 + 4p\right) + \left(-4p^3 + 6p^2 - 3p + 1\right). \tag{64}$$

B. CASE, $n_{\text{even}} = 4$

Let us now consider the correlated bit-flip noisy error model as defined in (43).

Encoding and Decoding Operators. Consider the following quantum code encoding 1 logical qubit into 4-physical qubits,

$$
\begin{aligned}
|0\rangle \;\rightarrow\; |0_L\rangle &\overset{\text{def}}{=} \frac{1}{\left(\sqrt{2}\right)^4}\left(|0\rangle_1 + |1\rangle_1\right)\otimes\left(|0\rangle_2 + |1\rangle_2\right)\otimes\left(|0\rangle_3 + |1\rangle_3\right)\otimes\left(|0\rangle_4 + |1\rangle_4\right) \equiv |++++\rangle\,,\\
|1\rangle \;\rightarrow\; |1_L\rangle &\overset{\text{def}}{=} \frac{1}{\left(\sqrt{2}\right)^4}\left(|0\rangle_1 - |1\rangle_1\right)\otimes\left(|0\rangle_2 - |1\rangle_2\right)\otimes\left(|0\rangle_3 - |1\rangle_3\right)\otimes\left(|0\rangle_4 + |1\rangle_4\right) \equiv |----\rangle\,,
\end{aligned} \tag{65}
$$

with $\langle ++++\,|\,++++\rangle = \langle ----\,|\,----\rangle = 1$ and $\langle ----\,|\,++++\rangle = \langle ++++\,|\,----\rangle = 0$. Following the line of reasoning used for the odd case and omitting technical details that will appear in Appendix B, the entanglement fidelity $\mathcal{F}_{DFS}^{(4)}(\mu,\, p)$ becomes,

$$
\begin{aligned}
\mathcal{F}_{DFS}^{(4)}(\mu,\, p) \;=\;& \mu^3\left(-8p^4 + 16p^3 - 10p^2 + 2p\right) + \mu^2\left(24p^4 - 48p^3 + 28p^2 - 4p\right) + \\
& + \mu\left(-24p^4 + 48p^3 - 30p^2 + 6p\right) + \left(8p^4 - 16p^3 + 12p^2 - 4p + 1\right).
\end{aligned} \tag{66}
$$

In Figure 2, we plot $\mathcal{F}_{RC}^{(4)}(\mu,\, p)$ and $\mathcal{F}_{DFS}^{(4)}(\mu,\, p)$ for three values of the error probability $p = 0.45$, $p = 0.40$ and $p = 0.35$. Following the line of reasoning presented above, it can be shown that $\mathcal{F}_{DFS}^{(5)}(\mu,\, p)$ and $\mathcal{F}_{DFS}^{(6)}(\mu,\, p)$ are given by,

$$
\begin{aligned}
\mathcal{F}_{DFS}^{(5)}(\mu,\, p) \;=\;& \mu^4\left(-16p^5 + 40p^4 - 36p^3 + 14p^2 - 2p\right) + \mu^3\left(64p^5 - 160p^4 + 136p^3 - 44p^2 + 4p\right) + \\
& + \mu^2\left(-96p^5 + 240p^4 - 204p^3 + 66p^2 - 6p\right) + \mu\left(64p^5 - 160p^4 + 144p^3 - 56p^2 + 8p\right) + \\
& + \left(-16p^5 + 40p^4 - 40p^3 + 20p^2 - 5p + 1\right),
\end{aligned} \tag{67}
$$

and,

$$
\begin{aligned}
\mathcal{F}_{DFS}^{(6)}(\mu,\, p) \;=\;& \mu^5\left(-32p^6 + 97p^5 - 115p^4 + 67p^3 - 19p^2 + 2p\right) + \\
& + \mu^4\left(160p^6 - 484p^5 + 546p^4 - 280p^3 + 62p^2 - 4p\right) + \\
& + \mu^3\left(-320p^6 + 966p^5 - 1068p^4 + 519p^3 - 103p^2 + 6p\right) + \\
& + \mu^2\left(320p^6 - 964p^5 + 1078p^4 - 546p^3 + 120p^2 - 8p\right) + \\
& + \mu\left(-160p^6 + 481p^5 - 561p^4 + 320p^3 - 90p^2 + 10p\right) + \\
& + \left(32p^6 - 96p^5 + 120p^4 - 80p^3 + 30p^2 - 6p + 1\right).
\end{aligned} \tag{68}
$$

respectively. It turns out that $\mathcal{F}_{DFS}^{(5)}(\mu,\, p) \le \mathcal{F}_{DFS}^{(3)}(\mu,\, p)$ and $\mathcal{F}_{DFS}^{(6)}(\mu,\, p) \le \mathcal{F}_{DFS}^{(4)}(\mu,\, p)$ for $\mu \in [0,\, 1]$ and $p < 0.5$. Moreover, $\mathcal{F}_{DFS}^{(4)}(\mu,\, p)$ is greater than $\mathcal{F}_{DFS}^{(3)}(\mu,\, p)$ for $\mu \ge \mu_{\min} \ge 0.2$. Therefore, in the high correlation regime where $\mu \rightarrow 1$, $\mathcal{F}_{\text{DFS}}^{(4)}(\mu,\, p)$ achieves the highest value for arbitrary error probabilities p less than 0.5.

V. FINAL REMARKS

Because of the results obtained in the previous Section, it follows that there must be a certain threshold value $\mu^*(p)$ that allows to select the better code between the repetition and the noiseless quantum code for our noisy quantum

memory channel. Considering the case with $n=4$, we may obtain a curve $\mu^{*}=\mu^{*}(p)$ defined in such a way that, $\mathcal{F}_{DFS}^{(4)}(\mu^{*}(p), p)-\mathcal{F}_{RC}^{(4)}(\mu^{*}(p), p)=0$. For example, In Figure 2 we have plotted $\mathcal{F}_{RC}^{(4)}(\mu, p)$ and $\mathcal{F}_{DFS}^{(4)}(\mu, p)$ for few values of p. From this plot, we see the emergence of threshold values $\mu^{*}(0.45)\simeq 0.34$, $\mu^{*}(0.40)\simeq 0.45$ and $\mu^{*}(0.35)\simeq 0.52$ when the curves $\mathcal{F}_{RC}^{(4)}(\mu, p)$ and $\mathcal{F}_{DFS}^{(4)}(\mu, p)$ cross. This means that for $p=0.45$ the noiseless quantum code outperforms the repetition code when $\mu\geq\mu^{*}(0.45)\simeq 0.34$. Finally, in Figure 3 we plot the threshold curve $\mu^{*}(p)$ for all permitted values of the error probability p. In conclusion, we have shown in an explicit way that the repetition code (be it even or odd) works better than the noiseless quantum code in the low correlations regime. On the contrary, in the high correlation regime, the noiseless quantum codes work better. The proper quantities defining the correlation regimes are the threshold values $\mu^{*}(p)$.

In conclusion, in this Letter we have analyzed the performance of simple quantum error correcting codes in the presence of correlated noise error models characterized by a correlation strength μ. Specifically, we have considered bit flip (phase flip) noisy quantum memory channels and used repetition and noiseless quantum codes. We have characterized the performance of the codes by means of the entanglement fidelity $\mathcal{F}(\mu, p)$ as function of the error probability p and degree of memory μ. We have shown in an explicit way that the entanglement fidelity $\mathcal{F}_{RC}^{(n_{odd})}(\mu, p)$ equals $\mathcal{F}_{RC}^{(n_{odd}+1)}(\mu, p)$ and that $\mathcal{F}_{RC}^{(n)}(\mu, p)$ increases with the length n of the code and decreases with the correlation parameter μ. Furthermore, we also used the decoherence free subspaces formalism and showed that the performance of such QECCs quantified in terms of the entanglement fidelity $\mathcal{F}_{DFS}^{(n)}(\mu, p)$ is better than the one of repetition codes in the high correlation regime where $\mu\rightarrow 1$. The noiseless quantum code with $n=4$ preforms better than the other (noiseless) codes considered in this work in the high correlation regime. Comparing the entanglement fidelities of repetition codes and noiseless quantum codes, we found a threshold $\mu^{*}(p)$ for the correlation strength that allows to select the quantum code with better performance.

The above results suggest that it may be convenient to concatenate decoherence-free subspaces with standard quantum error correcting codes in order to achieve higher entanglement fidelity values in both low and high correlations regimes. This will be the object of future investigations.

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Appendix A: Repetition Codes, $n_{\mathbf{even}}=4$

Recovery Operators. The set of error operators satisfying the detectability condition $P_{\mathcal{C}}A_{k}^{\prime}P_{\mathcal{C}}=\lambda_{A_{k}^{\prime}}P_{\mathcal{C}}$ where $P_{\mathcal{C}}=\left|0_{L}\right\rangle\left\langle 0_{L}\right|+\left|1_{L}\right\rangle\left\langle 1_{L}\right|$ is the projector operator on the code subspace $\mathcal{C}=Span\left\{\left|0_{L}\right\rangle,\ \left|1_{L}\right\rangle\right\}$ with $\left|0_{L}\right\rangle\overset{\text{def}}{=}\left|0000\right\rangle$ and $\left|1_{L}\right\rangle\overset{\text{def}}{=}\left|1111\right\rangle$ is given by $\mathcal{A}_{\text{detectable}}=\mathcal{A}\backslash\left\{A_{15}^{\prime}\right\}\subseteq\mathcal{A}$. Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $\mathcal{A}_{\text{correctable}}^{\dagger}\mathcal{A}_{\text{correctable}}$ is detectable. It follows then that,

$$\mathcal{A}_{\text{correctable}}=\left\{A_{0}^{\prime},\ A_{1}^{\prime},\ A_{2}^{\prime},\ A_{3}^{\prime},\ A_{4}^{\prime},\ A_{5}^{\prime},\ A_{6}^{\prime},\ A_{7}^{\prime}\right\}\subseteq\mathcal{A}_{\text{detectable}}\subseteq\mathcal{A}. \tag{A1}$$

The action of the correctable error operators $\mathcal{A}_{\text{correctable}}$ on the codewords $\left|0_{L}\right\rangle$ and $\left|1_{L}\right\rangle$ is given by,

$$\begin{aligned}\left|0_{L}\right\rangle &\rightarrow A_{0}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{0}^{(4)}}\left|0000\right\rangle,\ A_{1}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{1}^{(4)}}\left|1000\right\rangle,\ A_{2}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{2}^{(4)}}\left|0100\right\rangle,\ A_{3}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{3}^{(4)}}\left|0010\right\rangle,\\ A_{4}^{\prime}\left|0_{L}\right\rangle &= \sqrt{\tilde{p}_{4}^{(4)}}\left|0001\right\rangle,\ A_{5}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{5}^{(4)}}\left|1100\right\rangle,\ A_{6}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{6}^{(4)}}\left|1010\right\rangle,\ A_{7}^{\prime}\left|0_{L}\right\rangle=\sqrt{\tilde{p}_{7}^{(4)}}\left|1001\right\rangle,\end{aligned} \tag{A2}$$

and,

$$\begin{aligned}\left|1_{L}\right\rangle &\rightarrow A_{0}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{0}^{(4)}}\left|1_{L}\right\rangle,\ A_{1}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{1}^{(4)}}\left|0111\right\rangle,\ A_{2}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{2}^{(4)}}\left|1011\right\rangle,\ A_{3}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{3}^{(4)}}\left|1101\right\rangle,\\ A_{4}^{\prime}\left|1_{L}\right\rangle &= \sqrt{\tilde{p}_{4}^{(4)}}\left|1110\right\rangle,\ A_{5}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{5}^{(4)}}\left|0011\right\rangle,\ A_{6}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{6}^{(4)}}\left|0101\right\rangle,\ A_{7}^{\prime}\left|1_{L}\right\rangle=\sqrt{\tilde{p}_{7}^{(4)}}\left|0110\right\rangle,\end{aligned} \tag{A3}$$

respectively. The two eight-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_2^4$ generated by the action of $\mathcal{A}_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$\mathcal{V}^{0_L} = Span \left\{ \begin{array}{c} \left|v_1^{0_L}\right\rangle = |0000\rangle, \left|v_2^{0_L}\right\rangle = |1000\rangle, \ \left|v_3^{0_L}\right\rangle = |0100\rangle, \ \left|v_4^{0_L}\right\rangle = |0010\rangle, \\ \left|v_5^{0_L}\right\rangle = |0001\rangle, \left|v_6^{0_L}\right\rangle = |1100\rangle, \ \left|v_7^{0_L}\right\rangle = |1010\rangle, \ \left|v_8^{0_L}\right\rangle = |1001\rangle, \end{array} \right\}, \tag{A4}$$

and,

$$\mathcal{V}^{1_L} = Span \left\{ \begin{array}{c} \left|v_1^{1_L}\right\rangle = |1111\rangle, \left|v_2^{1_L}\right\rangle = |0111\rangle, \ \left|v_3^{1_L}\right\rangle = |1011\rangle, \ \left|v_4^{1_L}\right\rangle = |1101\rangle, \\ \left|v_5^{1_L}\right\rangle = |1110\rangle, \left|v_6^{1_L}\right\rangle = |0011\rangle, \ \left|v_7^{1_L}\right\rangle = |0101\rangle, \ \left|v_8^{1_L}\right\rangle = |0110\rangle \end{array} \right\}. \tag{A5}$$

Notice that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} = \mathcal{H}_2^4$. The recovery superoperator $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1,.., 8$ is defined as,

$$R_l \stackrel{\text{def}}{=} V_l \sum_{i=0}^{1} \left|v_l^{i_L}\right\rangle \left\langle v_l^{i_L}\right|, \tag{A6}$$

where the unitary operator V_l is such that $V_l \left|v_l^{i_L}\right\rangle = |i_L\rangle$ for $i \in \{0, 1\}$. Substituting (A4) and (A5) into (A6), it follows that the eight recovery operators $\{R_1, .., R_8\}$ are given by,

$$\begin{aligned} R_1 &= |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|, \ R_2 = |0_L\rangle\langle 1000| + |1_L\rangle\langle 0111|, \ R_3 = |0_L\rangle\langle 0100| + |1_L\rangle\langle 1011|, \\ R_4 &= |0_L\rangle\langle 0010| + |1_L\rangle\langle 1101|, \ R_5 = |0_L\rangle\langle 0001| + |1_L\rangle\langle 1110|, \ R_6 = |0_L\rangle\langle 1100| + |1_L\rangle\langle 0011| \\ R_7 &= |0_L\rangle\langle 1010| + |1_L\rangle\langle 0101|, \ R_8 = |0_L\rangle\langle 1001| + |1_L\rangle\langle 0110|. \end{aligned} \tag{A7}$$

It can be shown that $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1,.., 8$ is indeed a trace preserving quantum operation since,

$$\sum_{l=1}^{8} R_l^\dagger R_l = I_{16\times 16}. \tag{A8}$$

Considering this recovery operation $\mathcal{R}$, the map $\Lambda^{(4)}(\rho)$ in (43) becomes,

$$\Lambda_{\text{recover}}^{(4)}(\rho) \equiv \left(\mathcal{R}\circ\Lambda^{(4)}\right)(\rho) \stackrel{\text{def}}{=} \sum_{k=0}^{15}\sum_{l=1}^{8} \left(R_l A_k'\right)\rho\left(R_l A_k'\right)^\dagger. \tag{A9}$$

Entanglement Fidelity. We want to describe the action of $\mathcal{R}\circ\Lambda^{(4)}$ restricted to the code subspace $\mathcal{C}$. We simply compute the 2×2 matrix representation $[R_l A_k']_{|\mathcal{C}}$ of each $R_l A_k'$ with $l = 1,.., 4$ and $k = 0,.., 7$ where,

$$[R_l A_k']_{|\mathcal{C}} \stackrel{\text{def}}{=} \begin{pmatrix} \langle 0_L|R_l A_k'|0_L\rangle & \langle 0_L|R_l A_k'|1_L\rangle \\ \langle 1_L|R_l A_k'|0_L\rangle & \langle 1_L|R_l A_k'|1_L\rangle \end{pmatrix}. \tag{A10}$$

Substituting (A2), (A3) and (A7) into (A10), it turns out that the only matrices $[R_l A_k']_{|\mathcal{C}}$ with non-vanishing trace are given by,

$$\begin{aligned} [R_1 A_0']_{|\mathcal{C}} &= \sqrt{\tilde{p}_0^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ [R_2 A_1']_{|\mathcal{C}} = \sqrt{\tilde{p}_1^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ [R_3 A_2']_{|\mathcal{C}} = \sqrt{\tilde{p}_2^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ [R_4 A_3']_{|\mathcal{C}} &= \sqrt{\tilde{p}_3^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ [R_5 A_4']_{|\mathcal{C}} = \sqrt{\tilde{p}_4^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ [R_6 A_5']_{|\mathcal{C}} = \sqrt{\tilde{p}_5^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ [R_7 A_6']_{|\mathcal{C}} &= \sqrt{\tilde{p}_6^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ [R_8 A_7']_{|\mathcal{C}} = \sqrt{\tilde{p}_7^{(4)}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{A11}$$

Therefore, the entanglement fidelity $\mathcal{F}_{RC}^{(4)}(\mu, p)$ defined as,

$$\mathcal{F}_{RC}^{(4)}(\mu, p) \stackrel{\text{def}}{=} \mathcal{F}_{RC}^{(4)}\left(\frac{1}{2}I_{2\times 2}, \mathcal{R}\circ\Lambda^{(4)}\right) = \frac{1}{(2)^2}\sum_{k=0}^{15}\sum_{l=1}^{8}\left|\text{tr}\left([R_l A_k']_{|\mathcal{C}}\right)\right|^2, \tag{A12}$$

results,

$$\mathcal{F}_{RC}^{(4)}(\mu,\, p) = \tilde{p}_0^{(4)} + \tilde{p}_1^{(4)} + \tilde{p}_2^{(4)} + \tilde{p}_3^{(4)} + \tilde{p}_4^{(4)} + \tilde{p}_5^{(4)} + \tilde{p}_6^{(4)} + \tilde{p}_7^{(4)}. \tag{A13}$$

Substituting (19) into (A13), we finally obtain

$$\mathcal{F}_{RC}^{(4)}(\mu,\, p) = \mu^2\left(2p^3 - 3p^2 + p\right) + \mu\left(-4p^3 + 6p^2 - 2p\right) + \left(2p^3 - 3p^2 + 1\right). \tag{A14}$$

Notice that $\mathcal{F}_{RC}^{(4)}(\mu,\, p) = \mathcal{F}_{RC}^{(3)}(\mu,\, p)$ and, in absence of correlations,

$$\mathcal{F}_{RC}^{(4)}(0,\, p) = \sum_{m=0}^{1}\binom{4}{m}p^m(1-p)^{4-m} + \frac{1}{2}\binom{4}{2}p^2(1-p)^2 = 2p^3 - 3p^2 + 1 \equiv \mathcal{F}_{RC}^{(3)}(0,\, p). \tag{A15}$$

Finally, following the same line of reasoning presented above, it can be shown that $\mathcal{F}_{RC}^{(6)}(\mu,\, p) = \mathcal{F}_{RC}^{(5)}(\mu,\, p)$ and $\mathcal{F}_{RC}^{(8)}(\mu,\, p) = \mathcal{F}_{RC}^{(7)}(\mu,\, p)$.

Appendix B: Decoherence Free Subspaces, $n_{\text{even}} = 4$

Recovery Operators. The set of error operators satisfying the detectability condition $P_{\mathcal{C}}A'_kP_{\mathcal{C}} = \lambda_{A'_k}P_{\mathcal{C}}$ where $P_{\mathcal{C}} = |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projector operator on the code subspace $\mathcal{C} = Span\{|0_L\rangle,\ |1_L\rangle\}$ is given by,

$$\mathcal{A}_{\text{detectable}} = \{A'_0,\, A'_5,\, A'_6,\, A'_7,\, A'_8,\, A'_9,\, A'_{10},\, A'_{15}\} \subseteq \mathcal{A}. \tag{B1}$$

Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $\mathcal{A}^{\dagger}_{\text{correctable}}\mathcal{A}_{\text{correctable}}$ is detectable. It follows then that,

$$\mathcal{A}_{\text{correctable}} = \{A'_0,\, A'_5,\, A'_6,\, A'_7,\, A'_8,\, A'_9,\, A'_{10},\, A'_{15}\} \equiv \mathcal{A}_{\text{detectable}}. \tag{B2}$$

The action of the correctable error operators $\mathcal{A}_{\text{correctable}}$ on the codewords $|0_L\rangle$ and $|1_L\rangle$ is given by,

$$|0_L\rangle \to A'_{0r}|0_L\rangle = \sqrt{\tilde{p}_r^{(4)}}\,|0_L\rangle\,,\ |1_L\rangle \to A'_r|1_L\rangle = \sqrt{\tilde{p}_r^{(4)}}\,|1_L\rangle\,, \tag{B3}$$

for $r = 0,\, 5,\, 6,\, 7,\, 8,\, 9,\, 10,\, 15$. From (B3), it follows that $\mathcal{C} = Span\{|0_L\rangle,\ |1_L\rangle\}$ is a decoherence-free subspace for the correctable error operators in $\mathcal{A}_{\text{correctable}}$. The two one-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_2^4$ generated by the action of $\mathcal{A}_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$\mathcal{V}^{0_L} = Span\left\{\left|v_1^{0_L}\right\rangle = |++++\rangle\right\}, \tag{B4}$$

and,

$$\mathcal{V}^{1_L} = Span\left\{\left|v_1^{1_L}\right\rangle = |----\rangle\right\}, \tag{B5}$$

respectively. Notice that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} \neq \mathcal{H}_2^4$. This means that the trace preserving recovery superoperator $\mathcal{R}$ is defined in terms of one standard recovery operator R_1 and by the projector $R_{\perp}$ onto the orthogonal complement of $\bigoplus_{i=0}^{1}\mathcal{V}^{i_L}$, i. e. the part of the Hilbert space $\mathcal{H}_2^4$ which is not reached by acting on the code $\mathcal{C}$ with the correctable error operators. In the case under consideration,

$$R_1 \overset{\text{def}}{=} |++++\rangle\langle ++++| + |----\rangle\langle ----|\,,\ R_{\perp} = \sum_{s=1}^{14}|r_s\rangle\langle r_s|\,, \tag{B6}$$

where $\{|r_s\rangle\}$ is an orthonormal basis for $\left(\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L}\right)^{\perp}$. It can be shown that $\mathcal{R} \leftrightarrow \{R_1,\, R_{\perp}\}$ is a trace preserving quantum operation,

$$R_1^{\dagger}R_1 + R_{\perp}^{\dagger}R_{\perp} = I_{16\times 16}. \tag{B7}$$

Considering this recovery operation $\mathcal{R}$ with $R_2 \equiv R_\perp$, the map $\Lambda^{(4)}(\rho)$ in (43) becomes,

$$\Lambda^{(4)}_{\text{recover}}(\rho) \equiv \left(\mathcal{R}\circ\Lambda^{(4)}\right)(\rho) \overset{\text{def}}{=} \sum_{k=0}^{15}\sum_{l=1}^{2}\left(R_l A'_k\right)\rho\left(R_l A'_k\right)^{\dagger}, \tag{B8}$$

Entanglement Fidelity. We want to describe the action of $\mathcal{R}\circ\Lambda^{(4)}$ restricted to the code subspace $\mathcal{C}$. Therefore, we compute the 2×2 matrix representation $[R_l A'_k]_{|\mathcal{C}}$ of each $R_l A'_k$ with $l = 1$, 2 and $k = 0$,.., 15 and it turns out that the only matrices $[R_l A'_k]_{|\mathcal{C}}$ with non-vanishing trace are given by,

$$[R_1 A'_r]_{|\mathcal{C}} = \sqrt{\tilde{p}^{(4)}_r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{B9}$$

for $r = 0$, 5, 6, 7, 8, 9, 10, 15. Therefore, the entanglement fidelity $\mathcal{F}^{(4)}_{DFS}(\mu, p)$ defined as,

$$\mathcal{F}^{(4)}_{DFS}(\mu, p) \overset{\text{def}}{=} \mathcal{F}^{(4)}_{DFS}\left(\frac{1}{2}I_{2\times 2}, \mathcal{R}\circ\Lambda^{(4)}\right) = \frac{1}{(2)^2}\sum_{k=0}^{15}\sum_{l=1}^{2}\left|\text{tr}\left([R_l A'_k]_{|\mathcal{C}}\right)\right|^2, \tag{B10}$$

is given by,

$$\mathcal{F}^{(4)}_{DFS}(\mu, p) = \tilde{p}^{(4)}_0 + \tilde{p}^{(4)}_5 + \tilde{p}^{(4)}_6 + \tilde{p}^{(4)}_7 + \tilde{p}^{(4)}_8 + \tilde{p}^{(4)}_9 + \tilde{p}^{(4)}_{10} + \tilde{p}^{(4)}_{15}. \tag{B11}$$

After some algebra, it follows that,

$$\begin{aligned}\mathcal{F}^{(4)}_{DFS}(\mu, p) \;=\;& \mu^3\left(-8p^4 + 16p^3 - 10p^2 + 2p\right) + \mu^2\left(24p^4 - 48p^3 + 28p^2 - 4p\right) + \\ & + \mu\left(-24p^4 + 48p^3 - 30p^2 + 6p\right) + \left(8p^4 - 16p^3 + 12p^2 - 4p + 1\right).\end{aligned} \tag{B12}$$

Finally, following the same line of reasoning presented above, $\mathcal{F}^{(5)}_{DFS}(\mu, p)$ and $\mathcal{F}^{(6)}_{DFS}(\mu, p)$ can be computed as well.

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